



# MATHEMATICS ADMISSIONS TEST

For candidates applying for Mathematics, Computer Science or one of their joint degrees at OXFORD UNIVERSITY and/or IMPERIAL COLLEGE LONDON

Wednesday 5 November 2014

Time Allowed: 2½ hours

Please complete the following details in BLOCK CAPITALS

Surname					
Other names					
Candidate Number	M				

This paper contains 7 questions of which you should attempt 5. There are directions throughout the paper as to which questions are appropriate for your course.

**A: Oxford Applicants:** if you are applying to Oxford for the degree course:

- Mathematics or Mathematics & Philosophy or Mathematics & Statistics, you should attempt Questions **1,2,3,4,5**.
- Mathematics & Computer Science, you should attempt Questions **1,2,3,5,6**.
- Computer Science or Computer Science & Philosophy, you should attempt **1,2,5,6,7**.

**Directions under A take priority over any directions in B which are relevant to you.**

**B: Imperial Applicants:** if you are applying to Imperial College for any of the Mathematics courses: Mathematics, Mathematics (Pure Mathematics), Mathematics with a Year in Europe, Mathematics with Applied Mathematics/Mathematical Physics, Mathematics with Mathematical Computation, Mathematics with Statistics, Mathematics with Statistics for Finance, Mathematics Optimisation and Statistics, you should attempt Questions **1,2,3,4,5**.

Further credit cannot be obtained by attempting extra questions.

Question 1 is a multiple choice question with ten parts. Marks are given solely for correct answers but any rough working should be shown in the space between parts. Answer Question 1 on the grid on Page 2. Each part is worth 4 marks.

Answers to questions 2-7 should be written in the space provided, continuing on to the blank pages at the end of this booklet if necessary. Each of Questions 2-7 is worth 15 marks.

**This page has deliberately been left blank**

# MATHEMATICS ADMISSIONS TEST

Wednesday 5 November 2014


Time Allowed: 2½ hours

Please complete these details below in block capitals.

Centre Number												
Candidate Number	<b>M</b>											
UCAS Number (if known)				-				-				
Date of Birth			-			-						

Please tick the appropriate box:

- I have attempted Questions **1,2,3,4,5**
- I have attempted Questions **1,2,3,5,6**
- I have attempted Questions **1,2,5,6,7**

 <b>Admissions Testing Service</b>	Administered on behalf of the University of Oxford by the Admissions Testing Service, part of Cambridge Assessment, a non-teaching department of the University of Cambridge.
---	---

FOR OFFICE USE ONLY

Q1	Q2	Q3	Q4	Q5	Q6	Q7

1. For **ALL APPLICANTS**.

For each part of the question on pages 3-7 you will be given **five** possible answers, just one of which is correct. Indicate for each part **A-J** which answer (a), (b), (c), (d), or (e) you think is correct with a tick (✓) in the corresponding column in the table below. *Please show any rough working in the space provided between the parts.*

	(a)	(b)	(c)	(d)	(e)
A					
B					
C					
D					
E					
F					
G					
H					
I					
J					

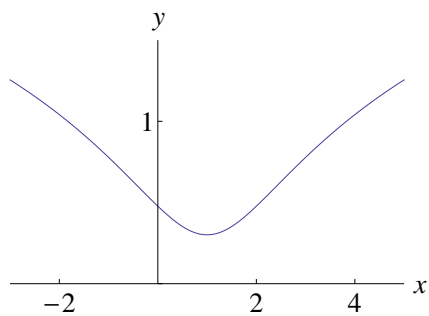
A. The inequality

$$x^4 < 8x^2 + 9$$

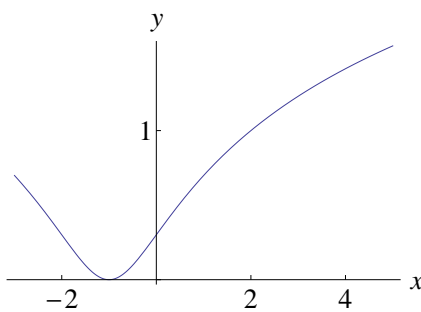
is satisfied precisely when

- (a)  $-3 < x < 3$ ; (b)  $0 < x < 4$ ; (c)  $1 < x < 3$ ; (d)  $-1 < x < 9$ ; (e)  $-3 < x < -1$ .

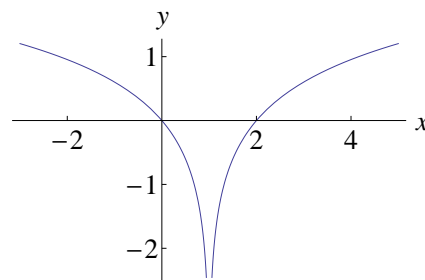
B. The graph of the function  $y = \log_{10}(x^2 - 2x + 2)$  is sketched in



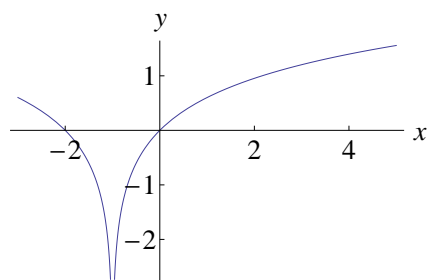
(a)



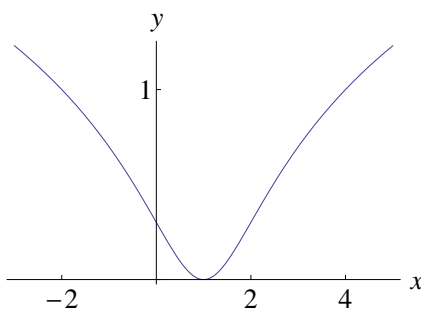
(b)



(c)



(d)



(e)

Turn over

C. The cubic

$$y = kx^3 - (k + 1)x^2 + (2 - k)x - k$$

has a turning point, that is a minimum, when  $x = 1$  precisely for

- (a)  $k > 0$ , (b)  $0 < k < 1$ , (c)  $k > \frac{1}{2}$ , (d)  $k < 3$ , (e) all values of  $k$ .

D. The reflection of the point  $(1, 0)$  in the line  $y = mx$  has coordinates

- (a)  $\left(\frac{m^2 + 1}{m^2 - 1}, \frac{m}{m^2 - 1}\right)$ , (b)  $(1, m)$ , (c)  $(1 - m, m)$ ,  
(d)  $\left(\frac{1 - m^2}{1 + m^2}, \frac{2m}{1 + m^2}\right)$ , (e)  $(1 - m^2, m)$ .

**E.** As  $x$  varies over the real numbers, the largest value taken by the function

$$(4 \sin^2 x + 4 \cos x + 1)^2$$

equals

- (a)  $17+12\sqrt{2}$ ,    (b)  $36$ ,    (c)  $48\sqrt{2}$ ,    (d)  $64-12\sqrt{3}$ ,    (e)  $81$ .

**F.** The functions  $S$  and  $T$  are defined for real numbers by

$$S(x) = x + 1, \quad \text{and} \quad T(x) = -x.$$

The function  $S$  is applied  $s$  times and the function  $T$  is applied  $t$  times, *in some order*, to produce the function

$$F(x) = 8 - x.$$

It is possible to deduce that:

- (a)  $s = 8$  and  $t = 1$ .
- (b)  $s$  is odd and  $t$  is even.
- (c)  $s$  is even and  $t$  is odd.
- (d)  $s$  and  $t$  are powers of 2.
- (e) none of the above.

Turn over

**G.** Let  $n$  be a positive integer. The coefficient of  $x^3y^5$  in the expansion of

$$(1 + xy + y^2)^n$$

equals

(a)  $n$ ,      (b)  $2^n$ ,      (c)  $\binom{n}{3}\binom{n}{5}$ ,      (d)  $4\binom{n}{4}$ ,      (e)  $\binom{n}{8}$ .

**H.** The function  $F(n)$  is defined for all positive integers as follows:  $F(1) = 0$  and for all  $n \geq 2$ ,

$F(n) = F(n - 1) + 2$	if 2 divides $n$ but 3 does not divide $n$ ;
$F(n) = F(n - 1) + 3$	if 3 divides $n$ but 2 does not divide $n$ ;
$F(n) = F(n - 1) + 4$	if 2 and 3 both divide $n$ ;
$F(n) = F(n - 1)$	if neither 2 nor 3 divides $n$ .

The value of  $F(6000)$  equals

(a) 9827,      (b) 10121,      (c) 11000,      (d) 12300,      (e) 12352.



**I.** The graph of the function

$$y = 2^{x^2-4x+3}$$

can be obtained from the graph of  $y = 2^{x^2}$  by

- (a) a stretch parallel to the  $y$ -axis followed by a translation parallel to the  $y$ -axis.
- (b) a translation parallel to the  $x$ -axis followed by a stretch parallel to the  $y$ -axis.
- (c) a translation parallel to the  $x$ -axis followed by a stretch parallel to the  $x$ -axis.
- (d) a translation parallel to the  $x$ -axis followed by reflection in the  $y$ -axis.
- (e) reflection in the  $y$ -axis followed by translation parallel to the  $y$ -axis.

**J.** For all real numbers  $x$ , the function  $f(x)$  satisfies

$$6 + f(x) = 2f(-x) + 3x^2 \left( \int_{-1}^1 f(t) dt \right).$$

It follows that  $\int_{-1}^1 f(x) dx$  equals

- (a) 4,      (b) 6,      (c) 11,      (d)  $\frac{27}{2}$ ,      (e) 23.

Turn over

**2. For ALL APPLICANTS.**

Let  $a$  and  $b$  be real numbers. Consider the cubic equation

$$x^3 + 2bx^2 - a^2x - b^2 = 0. \quad (*)$$

(i) Show that if  $x = 1$  is a solution of  $(*)$  then

$$1 - \sqrt{2} \leq b \leq 1 + \sqrt{2}.$$

(ii) Show that there is no value of  $b$  for which  $x = 1$  is a repeated root of  $(*)$ .

(iii) Given that  $x = 1$  is a solution, find the value of  $b$  for which  $(*)$  has a repeated root.

For this value of  $b$ , does the cubic

$$y = x^3 + 2bx^2 - a^2x - b^2$$

have a maximum or minimum at its repeated root?

Turn over

3.

For APPLICANTS IN  $\left\{ \begin{array}{l} \text{MATHEMATICS} \\ \text{MATHEMATICS \& STATISTICS} \\ \text{MATHEMATICS \& PHILOSOPHY} \\ \text{MATHEMATICS \& COMPUTER SCIENCE} \end{array} \right\}$  ONLY.

*Computer Science* and *Computer Science & Philosophy* applicants should turn to page 14.

The function  $f(x)$  is defined for all real numbers and has the following properties, valid for all  $x$  and  $y$ :

- (A)  $f(x + y) = f(x) f(y)$ .
- (B)  $df/dx = f(x)$ .
- (C)  $f(x) > 0$ .

Throughout this question, these should be the only properties of  $f$  that you use; no marks will be awarded for any use of the exponential function.

Let  $a = f(1)$ .

(i) Show that  $f(0) = 1$ .

(ii) Let

$$I = \int_0^1 f(x) dx.$$

Show that  $I = a - 1$ .

(iii) The trapezium rule with  $n$  steps is used to produce an estimate  $I_n$  for the integral  $I$ . Show that

$$I_n = \frac{1}{2n} \left( \frac{b+1}{b-1} \right) (a-1)$$

where  $b = f(1/n)$ .

(iv) Given that  $I_n \geq I$  for all  $n$ , show that

$$a \leq \left( 1 + \frac{2}{2n-1} \right)^n.$$

Turn over

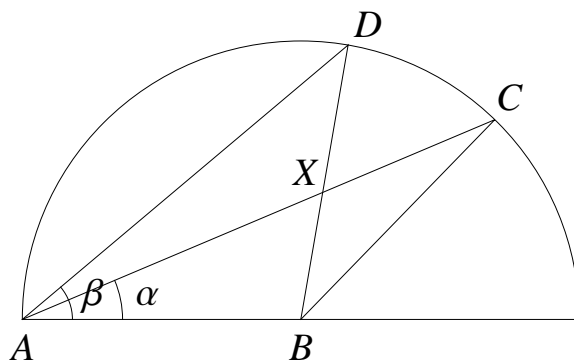
4.

For APPLICANTS IN  $\left\{ \begin{array}{l} \text{MATHEMATICS} \\ \text{MATHEMATICS \& STATISTICS} \\ \text{MATHEMATICS \& PHILOSOPHY} \end{array} \right\}$  ONLY.

*Mathematics & Computer Science, Computer Science and Computer Science & Philosophy* applicants should turn to page 14.

In the diagram below is sketched a semicircle with centre  $B$  and radius 1. Three points  $A, C, D$  lie on the semicircle as shown with  $\alpha$  denoting angle  $CAB$  and  $\beta$  denoting angle  $DAB$ . The triangles  $ABC$  and  $ABD$  intersect in a triangle  $ABX$ .

Throughout the question we shall consider the value of  $\alpha$  fixed. Assume for now that  $0 < \alpha \leq \beta \leq \pi/2$ .



(i) Show that the area of the triangle  $ABC$  equals

$$\frac{1}{2} \sin(2\alpha).$$

(ii) Let

$$F = \frac{\text{area of triangle } ABX}{\text{area of triangle } ABC}.$$

Without calculation, explain why, for every  $k$  in the range  $0 \leq k \leq 1$ , there is a unique value of  $\beta$  such that  $F = k$ .

(iii) Find the value of  $\beta$  such that  $F = 1/2$ .

(iv) Show that

$$F = \frac{\sin(2\beta) \sin \alpha}{\sin(2\beta - \alpha) \sin(2\alpha)}.$$

(v) Suppose now that  $0 < \beta < \alpha \leq \pi/2$ . Write down, without further calculation, an expression for the area of  $ABX$  and hence a formula for  $F$ .

Turn over

## 5. For ALL APPLICANTS.

Poets use *rhyme schemes* to describe which lines of a poem rhyme. Each line is denoted by a letter of the alphabet, with the same letter given to two lines that rhyme. To say that a poem has the rhyming scheme ABABCDED, indicates that the first and third lines rhyme, the second and fourth lines rhyme, and the sixth and eighth lines rhyme, but no others.

More precisely, the first line of the poem is given the letter A. If a subsequent line rhymes with an earlier line, it is given the same letter; otherwise, it is given the first unused letter. (For the purposes of this question, you can assume that we have an infinite supply of “letters”, not just the 26 letters of the alphabet.)

The purpose of this question is to investigate how many different rhyme schemes there are for poems of  $n$  lines. We write  $r_n$  for this number.

- (i) There are five different rhyming schemes for poems of three lines (so  $r_3 = 5$ ). List them.

Let  $c_{n,k}$  denote the number of rhyme schemes for poems with lines  $n$  that use exactly  $k$  different letters. For example  $c_{3,2} = 3$  corresponding to the rhyming schemes AAB, ABA and ABB.

- (ii) What is  $c_{n,1}$  for  $n \geq 1$ ?

What is  $c_{n,n}$ ?

Explain your answers.

- (iii) Suppose that  $1 < k < n$ . By considering the final letter of a rhyming scheme, explain why

$$c_{n,k} = kc_{n-1,k} + c_{n-1,k-1} .$$

- (iv) Write down an equation showing how to calculate  $r_n$  in terms of the  $c_{n,k}$ . Hence calculate  $r_4$ .
- (v) Give a formula for  $c_{n,2}$  in terms of  $n$  (for  $n \geq 2$ ). Justify your answer.



Turn over

6.

For APPLICANTS IN  $\left\{ \begin{array}{l} \text{COMPUTER SCIENCE} \\ \text{MATHEMATICS \& COMPUTER SCIENCE} \\ \text{COMPUTER SCIENCE \& PHILOSOPHY} \end{array} \right\}$  ONLY.

Alice plays a game 5 times with her friends Sam and Pam. In each game Alice chooses two integers  $x$  and  $y$  with  $1 \leq x \leq y$ . She whispers the sum  $x + y$  to Sam, and the product  $x \times y$  to Pam, so that neither knows what the other was told. Sam and Pam then have to try to work out what the numbers  $x$  and  $y$  are. Sam and Pam are well known expert logicians.

(i) In the first game, Pam says “I know  $x$  and  $y$ .”

What can we deduce about the values of  $x$  and  $y$ ? Explain your answer.

(ii) In the second game, Pam says “I don’t know what  $x$  and  $y$  are.”

Sam then says “I know  $x$  and  $y$ .”

Suppose the sum is 4. What are  $x$  and  $y$ ? Explain your answer.

(iii) In the third game, Pam says “I don’t know what  $x$  and  $y$  are.”

Sam then says “I don’t know what  $x$  and  $y$  are.”

Pam then says “I now know  $x$  and  $y$ .”

Suppose the product is 4. What are  $x$  and  $y$ ? Explain your answer.

(iv) In the fourth game, Pam says “I don’t know what  $x$  and  $y$  are.”

Sam then says “I already knew that.”

Pam then says “I now know  $x$  and  $y$ .”

Suppose the product is 8. What are  $x$  and  $y$ ? Explain your answer.

(v) Finally, in the fifth game, Pam says “I don’t know what  $x$  and  $y$  are.”

Sam then says “I don’t know what  $x$  and  $y$  are.”

Pam then says “I don’t know what  $x$  and  $y$  are.”

Sam then says “I now know  $x$  and  $y$ .”

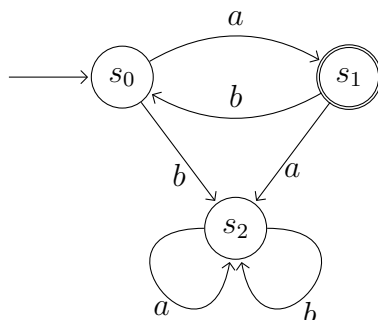
Suppose the sum is 5. What are  $x$  and  $y$ ? Explain your answer.

Turn over

7.

For APPLICANTS IN { COMPUTER SCIENCE  
COMPUTER SCIENCE & PHILOSOPHY } ONLY.

A *finite automaton* is a mathematical model of a simple computing device. A small finite automaton is illustrated below.



A finite automaton has some *finite* number of states; the above automaton has three states, labelled  $s_0$ ,  $s_1$  and  $s_2$ . The initial state,  $s_0$ , is indicated with an incoming arrow. The automaton receives *inputs* (e.g. via button presses), which might cause it to change state. In the example, the inputs are  $a$  and  $b$ . The state changes are illustrated by arrows; for example, if the automaton is in state  $s_1$  and it receives input  $b$ , it changes to state  $s_0$ ; if it is in state  $s_2$  and receives either input, it remains in state  $s_2$ . (For each state, there is precisely one out-going arrow for each input.)

Some of the states are defined to be *accepting states*; in the example, just  $s_1$  is defined to be an accepting state, represented by the double circle. A *word* is a sequence of inputs. The automaton *accepts* a word  $w$  if that sequence of inputs leads to an accepting state from the initial state. For example, the above automaton accepts the word  $aba$ .

- (i) Write down a description of the set of words accepted by the above automaton. A clear but informal description will suffice.
- (ii) Suppose we alter the above automaton by swapping accepting and non-accepting states; i.e. we make  $s_0$  and  $s_2$  accepting, and make  $s_1$  non-accepting. Write down a description of the set of words accepted by this new automaton. Again, a clear but informal description will suffice.
- (iii) Draw an automaton that accepts all words containing an even number (possibly zero) of  $a$ 's and any number of  $b$ 's (and no other words).
- (iv) Now draw an automaton that accepts all words containing an even number of  $a$ 's *or* an odd number of  $b$ 's (and no other words).

Let  $a^n$  represent  $n$  consecutive  $a$ 's. Let  $L$  be the set of all words of the form  $a^n b^n$  where  $n = 0, 1, 2, \dots$ ; i.e. all words composed of some number of  $a$ 's followed by the *same* number of  $b$ 's. We will show that no finite automaton accepts precisely this set of words.

- (v) Suppose a particular finite automaton  $A$  *does* accept precisely the words in  $L$ . Show that if  $i \neq j$  then the words  $a^i$  and  $a^j$  must lead to different states of  $A$ .

Hence show that this leads to a contradiction.

End of last question

This page has been intentionally left blank

This page has been intentionally left blank

This page has been intentionally left blank